Quadratization of higher degree binary optimization problems

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Joint work with Alexander Fix (Cornell), Aritanan Gruber (Rutgers), and Ramin Zabih (Cornell)
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- **Pseudo-Boolean Function**: $f : \{0, 1\}^n \to \mathbb{R}$

- **Multilinear Polynomial**: $f(x) = \sum_{S \subseteq [n]} \alpha_S \prod_{j \in S} x_j$

  Unique! (Hammer and Rudeanu, 1968)

- **PBO**:

  $$\min_{x \in \{0, 1\}^n} f(x)$$

  Very hard! (Grass, Tree and The Flowers, 1970+)

- **Many Applications**: ... computer vision!

- **QPBF**: $\alpha_S = 0$ for all $|S| \geq 3$.

- **QPBO**: efficient network flow based preprocessing

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- Given \( f : \{0, 1\}^n \to \mathbb{R} \) find quadratic \( g : \{0, 1\}^{n+m} \to \mathbb{R} \) such that
  \[
  f(x) = \min_{y \in \{0, 1\}^m} g(x, y) \quad \forall \ x \in \{0, 1\}^n.
  \]

- Keep \( m \) small!
- Have \( g \) as submodular as possible!
- Do not introduce large coefficients!
- Have it ALL!

Rosenberg, 1975: All PBFs have polynomial sized quadratizations.
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- A PBF \( f : \{0, 1\}^n \to \mathbb{R} \) is submodular if
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- Polynomial recognition if \( \deg(f) \leq 3 \).
  \( \text{(Billionnet and Minoux, 1985)} \)

- Recognition is NP-hard if \( \deg(f) \geq 4 \).
  \( \text{(Gallo and Simeone, 1989)} \)

- A QPBF is submodular iff it has no positive quadratic terms.
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- Which PBFs have submodular quadratization in poly time?

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\[ p(x, y, w) = xy - 2xw - 2yw + 3w = \begin{cases} 
= 0 & \text{if } w = xy, \\
\geq 1 & \text{if } w \neq xy 
\end{cases} \]

\[ f(x, y, \ldots) = xyA + B = \min_{w \in \{0, 1\}} wA + B + Mp(x, y, w) \]

if \( M \) is large enough.

- Many positive quadratic terms with large coefficients (recursion!), even if the input is submodular.
- NP-hard to find a quadratization in this way with the minimum number of new variables.
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- **Kolmogorov and Zabih (2004), Fredman and Drineas (2005):**
  \[-x_1x_2\cdots x_d = \min_{w \in \{0,1\}} w(d - 1 - x_1 - x_2 - \cdots - x_d)\]

- **Rother, Kohli, Feng and Jia (2009):**
  \[-\prod_{j \in N} \overline{x}_j \prod_{j \in P} x_j = \min_{u,v \in \{0,1\}} -uv + u \sum_{j \in N} x_j + v \sum_{j \in P} \overline{x}_j\]

- **Only one or two new variables per term; at most one positive quadratic term; no large coefficients.**

Theorem (vs. Billionet and Minoux (1985))

Cubic submodular functions have submodular quadratization of polynomial size with no large coefficients.
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- **Kolmogorov and Zabih (2004), Fredman and Drineas (2005):**

\[ -x_1 x_2 \cdots x_d = \min_{w \in \{0,1\}} w (d - 1 - x_1 - x_2 \cdots - x_d) \]

- **Rother, Kohli, Feng and Jia (2009):**

\[ -\prod_{j \in N} x_j \prod_{j \in P} x_j = \min_{u,v \in \{0,1\}} -uv + u \sum_{j \in N} x_j + v \sum_{j \in P} \overline{x}_j \]

- Only one or two new variables per term; at most one positive quadratic term; no large coefficients.

**Theorem (vs. Billionet and Minoux (1985)):**

*Cubic submodular functions have submodular quadratization of polynomial size with no large coefficients.*
Positive Terms

- **Ishikawa (2009, 2011):**

\[
\prod_{j=1}^{d} x_j = S_2(x) + \min_{w \in \{0,1\}^k} B(w) - 2A(w)S_1(x) + \rho [S_1(x) - d + 1]
\]

where \( d = 2k + 2 - \rho \), \( \rho \in \{0,1\} \), and

\[
S_1(x) = \sum_{j=1}^{d} x_j \quad \quad \quad \quad S_2(x) = \sum_{1 \leq i < j \leq d} x_i x_j
\]

\[
A(w) = \sum_{j=1}^{k} w_j \quad \quad \quad \quad B(w) = \sum_{j=1}^{k} (4j - 1)w_j
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- Only \( \approx d/2 \) new variables per term; no large coefficients; many positive quadratic terms.
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Multiple Splits

Assume that $\phi_i(w) \in \{0, 1\}$ for $i \in [q]$, $w \in \{0, 1\}^p$ such that

$$\min_{w \in \{0, 1\}^p} \sum_{i=1}^{q} \phi_i(w) = 1,$$

and

$$\forall I \subseteq [q] \exists w^* \in \{0, 1\}^p \text{ s.t. } \sum_{i \in I} \phi_i(w^*) = 0.$$

For instance $\phi_1 = w_1$, $\phi_2 = w_2$, and $\phi_3 = \bar{w}_1 \bar{w}_2$ is such a system.

Theorem

If $P_i$, $i \in [q]$ are subsets of indices covering $[d]$, then we have

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With $k = \lceil \log q \rceil$ new variables we can split a degree $d = pq$ term into $q$ terms of degree $k + p$. 
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Let $C \subseteq [n]$, $\mathcal{H} \subseteq 2^{[n]} \setminus C$, and consider the following fragment of a pseudo-Boolean function:

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where $\alpha_H \geq 0$ for all $H \in \mathcal{H}$.

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$$g(x) = \min_{w \in \{0,1\}} \left( \sum_{H \in \mathcal{H}} \alpha_H \right) w \prod_{j \in C} x_j + \sum_{H \in \mathcal{H}} \alpha_H \overline{w} \prod_{j \in H} x_j.$$

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Corollary

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Ishikawa’s method provides a quadratization with $\approx n + \frac{td}{2}$ new variables and $\max\{\binom{n}{2}, t\binom{d}{2}\}$ positive quadratic terms.

<table>
<thead>
<tr>
<th></th>
<th>New variables</th>
<th># positive terms</th>
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<tbody>
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<td>Ishikawa</td>
<td>224,346</td>
<td>421,897</td>
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Figure: Performance comparison of reductions, on Ishikawa’s benchmarks. Relative performance of our method is shown as $\Delta$. (Joint work with Alexander Fix and Ramin Zabih (Cornell University).)
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